

Algebraic Aspects of the Chirality Phenomenon in Chemistry

ERNST RUCH

Institut für Quantenchemie der Freien Universität, Berlin, Germany

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Chirality is the property which differentiates an object from its mirror image. It is manifested in experiments which lead to the determination of a real number, depending on the measured object but not its orientation in space, and having opposite signs for antipodic objects.

We may refer to measurements, such as optical rotation or circular dichroism, as "pseudoscalar measurements." In chemistry, pseudoscalar measurements are used to get information about the structure of molecules. For this purpose one needs aspects to relate the measured values to the influence of some individual parts of a molecule, their interactions, and their location within a molecular skeleton.

There are many theories explaining special pseudoscalar measurements of special molecules, partly on quantum-mechanical, partly on classical grounds, but no effort has been made to establish a general analysis on an algebraic basis. An algebraic treatment, however, is just what is required to provide qualitative concepts which give insight into the phenomenon itself as well as into the structure of relevant theories whatever kind they may be. The purpose of this Account is to review some results which are part of a general algebraic theory derived in a series of previous papers.¹⁻⁴ We restrict ourselves to a selection of those statements which can be formulated without going too far into mathematical details.

Homochirality³

If asked to put our left shoes into one box and our right shoes into a second box we could accomplish the task without mental difficulty, in spite of the fact that the right shoes belonging to different people may be quite different in color, shape, and size and although, probably, there is not a single pair of shoes which are precise mirror images of each other.

If asked to solve the same problem with potatoes, we must capitulate. Of course, it is possible that by chance we find an antipodal pair. It is then clear that we must separate them, but for other potatoes different in shape, we have to make new arbitrary decisions each time. Any classification would be very artificial.

Ernst Ruch was born in Munich in 1919. He started his higher education in engineering at the Technical University in Munich, went on to theoretical physics, and took a postgraduate position in mathematics there. His first activity in chemistry was working out the theory of binding in sandwich compounds, a problem with which he became acquainted through a classmate, E. O. Fischer, who was working on their synthesis. Professor Ruch's interest in stereochemistry became the starting point for further purely mathematical investigations. Since 1966, he has been the Director of the Institute for Quantum Chemistry at the Free University in Berlin.

In order to put the problem into manageable form, we require well-defined sets of objects—let us say classes of molecules, because molecules are what we want to classify. Molecules belonging to a given class may be characterized by a common achiral molecular skeleton with ligands attached at the skeleton sites. We restrict ourselves to ligands which fulfill the condition that molecules have the symmetry of the bare skeleton if all its ligands are of the same kind. Therefore molecules containing only ligands of one sort are achiral. But, if ligands of different kinds are attached, we generally have chiral molecules, provided we are not considering special skeletons, for instance, skeletons in which all ligand sites lie in one plane of symmetry, *e.g.*, benzene.

We will discuss chiral classes which are specified by skeletons such that molecules are chiral, at least if all ligands are different. We assume that the ligands may be described by a one-parametric quality, measured by the parameter λ , such that ligands of the same kind have equal parameter values. For a simple visualization we may take the ligands to be spheres of different size, the diameters being the parameters. It must be emphasized that this is an unnecessary limitation to a special picture. Examples of such skeletons are all achiral arrangements of a given number of sites in space, *e.g.*, the corners of regular bodies (see Figure 1). Now we consider the whole set of molecules belonging to a class in the sense defined above and ask the question: what are the necessary and sufficient conditions for a class which permit us to separate all chiral molecules within that class into two antipodal subclasses such that the classification principle is as obvious as in the case of shoes? In other words, when is it possible to apply a concept like chiral relatedness which tells us for each pair of molecules whether or not they are equivalent (*i.e.*, whether or not they have the property in common that we are seeking)? The classification principle must not depend on arguments other than those inherent in chirality. If such a concept exists, we call all molecules belonging to one subclass homochiral and molecules belonging to different subclasses heterochiral.

Chiral relatedness, *i.e.*, chiral similarity of molecules within a class in question, certainly has to be based on similarity of ligands. Because a small change in the

(1) E. Ruch, A. Schönhofer, and I. Ugi, *Theor. Chim. Acta*, **7**, 420 (1967).

(2) E. Ruch and A. Schönhofer, *ibid.*, **10**, 91 (1968).

(3) E. Ruch, *ibid.*, **11**, 183, 462 (1968).

(4) E. Ruch and A. Schönhofer, *ibid.*, **19**, 225 (1970).

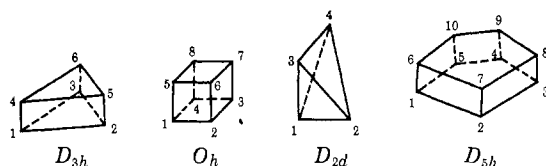


Figure 1.

ligand qualities gives rise to small changes in the values of continuous functions of the ligand parameters, relatedness may be discussed in terms of continuous functions. Continuous pseudoscalar functions consequently express similarity of molecules by neighboring values and distinguish enantiomers by the sign. Any separation into enantiomeric subclasses containing molecules, which may be regarded as similar, can be related to the sign of a properly chosen continuous pseudoscalar function. But this classification is as arbitrary as the choice of this function. The homochirality concept, however, does not allow this arbitrariness. From this we are forced to the conclusion that a homochirality concept necessitates the existence of a special continuous pseudoscalar function, which has only "achiral zeros," *i.e.*, which has no zeros except for achiral molecules. Hence we understand the significance of a mathematical fact proved elsewhere.³

Continuous pseudoscalar functions without chiral zeros do exist, but only for certain classes. Classes for which such functions exist shall be called classes of category a; otherwise, classes of category b. The result of the mathematical analyses can be stated as follows: A class belongs to category a if, and only if, either the skeleton has only two sites for ligands or the number n of sites is larger, but the symmetry of the skeleton contains mirror planes and each mirror plane contains $n - 2$ sites (sites which are fix-points for all symmetry operations of the skeleton are not taken into account). Accordingly, all the other classes belong to category b. Two examples may explain the situation (see Figure 2). It is clear from the above definition that the trigonal bipyramid belongs to category a and the tetragonal bipyramid belongs to category b.

The lack of a homochirality concept for category b can be shown explicitly for the tetragonal bipyramid by following a simple argument which reveals, to a certain degree, the idea of the general mathematical proof of the above theorem.

We vary the ligands at positions 1 and 2 continuously so that we end with an interchanged position of the original ligands. Because we do not encounter an achiral situation in the course of this variation we may assign any pair of neighboring molecules to one of the two enantiomeric subclasses. Hence all molecules which we were passing on this path should be assigned to the same subclass. Following this, we perform the same variation with the ligands at positions 3 and 4. The same argument applies. But now we have got the enantiomer of the original molecule, and this one must certainly be in the enantiomeric subclass. Therefore,

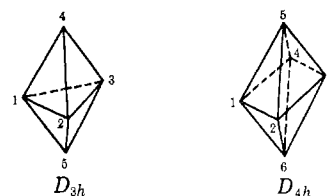


Figure 2.

chiral relatedness between neighboring molecules must be interrupted somewhere on the path from the original molecule to its enantiomer. But where? No privileged point can be found; one would have to make an arbitrary decision. Therefore we have to conclude that a homochirality concept does not exist for our class, and this fact is in agreement with our general statement.

Remembering our theorem that continuous pseudoscalar functions without chiral zeros do exist for classes belonging to category a, we can use the sign of its values as the criterion for homochiral molecules. It can be proved that pseudoscalar polynomials of the lowest degree in ligand-specific parameters λ are functions of this sort. For the classes of category a, and for these alone, the mentioned polynomials have a special form which we call "chirality product." It is a product of differences of two parameters in all cases. For instance, the polynomial for the trigonal bipyramid is

$$\chi(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5) = (\lambda_1 - \lambda_2)(\lambda_2 - \lambda_3)(\lambda_3 - \lambda_1)(\lambda_4 - \lambda_5)$$

where the indices refer to the ligands on the correspondingly numbered skeleton sites. A general simple technique for finding these polynomials has been developed.² We may call all molecules for which the polynomial is positive "right," and all molecules for which the polynomial is negative "left," or *vice versa*.

If our definition is physically relevant it must refer to the determining qualities of the ligands measured by the parameters λ_i and must not depend on the special parameter chosen. This means, while changing the parameters, but keeping parameters which measure the same quality, the definition for chiral relatedness should not be affected. Parameters describing the same quality are monotonic functions of each other: $\lambda' = \lambda'(\lambda)$, with the condition $d\lambda'/d\lambda > 0$. In this case, from $\lambda_i < \lambda_k$ follows $\lambda'(\lambda_i) < \lambda'(\lambda_k)$ and, therefore, the chirality product of the parameters λ_i and that of $\lambda'_i = \lambda'(\lambda_i)$ has the same sign for each molecule. Hence we see that the classification principle does not depend on the chosen parameter but only on the responsible physical quality of the ligands. In order to find the sign of the chirality product we need only the sequence of the parametric values for different ligands. This statement does not apply to a classification according to enantiomeric subclasses when concerned with case b, which again shows that a classification would then always be very artificial.

A nomenclature such as the *RS* nomenclature pro-

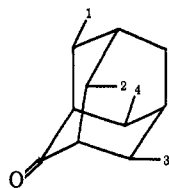


Figure 3.

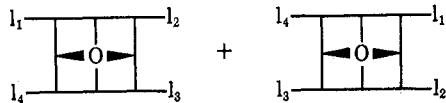


Figure 4.

posed by Cahn, Ingold, and Prelog which refers to the weight of ligands can be defined to give a classification according to homochirality for classes of category a. For b classes, on the other hand, it remains just a nomenclature, necessary and useful, but without physical meaning, like the concepts right and left for potatoes.

Qualitative Completeness and Ensemble Operators

Whether we describe a chirality measurement by a conventional theory based on a classical or a quantum-mechanical treatment or whether we start with chirality functions of a certain structure given by mathematical simplicity and having certain degrees of freedom adjustable to the special kind of measurements,² we are confronted with a phenomenon which shows that a general criticism applies.⁴

Let us study this on the basis of the well-known sector rules applied to the Cotton effect in adamantane derivatives (Figure 3). Our first method, the method of polynomials of lowest degree in ligand-specific parameters,^{2,4} leads to the chirality function

$$\chi(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = \lambda_1 - \lambda_2 + \lambda_3 - \lambda_4$$

which is equivalent to the quadrant rule. Now let us apply this method for the nonracemic isomeric mixture of Figure 4. The superposition of the two responsible polynomials gives

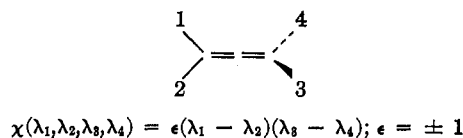
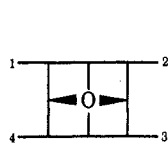
$$\lambda_1 - \lambda_2 + \lambda_3 - \lambda_4 + \lambda_4 - \lambda_1 + \lambda_2 - \lambda_3 \equiv 0$$

That shows that the quadrant rule as well as our Ansatz gives zero for a nonracemic mixture of isomers, whatever the nature of the ligands may be. This is a fact that demonstrates that the chirality function as well as the quadrant rule is missing an essential point in principle.

The same situation is met in the case of allene derivatives (Figure 5) where we find a nonracemic mixture (see Figure 6) for which we get the identity

$$\frac{1}{3} \{ \chi(\lambda_1, \lambda_2, \lambda_3, \lambda_4) + \chi(\lambda_1, \lambda_4, \lambda_2, \lambda_3) + \chi(\lambda_1, \lambda_3, \lambda_4, \lambda_2) \} \equiv 0$$

The analogous phenomenon is normally found whatever the classes are and the theories concerned. In many cases one finds even chiral molecules with partly



$$\chi(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = \epsilon(\lambda_1 - \lambda_2)(\lambda_3 - \lambda_4); \epsilon = \pm 1$$

Figure 5.

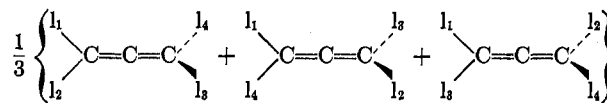


Figure 6.

equal ligands for which the chirality function vanishes independently of the nature of these ligands, but then always a nonracemic mixture of isomers with ligands different from each other can be found for which an identity relation of the above type holds. Therefore what we have to demand from a satisfactory theory is that it leads to a chirality function for which the following statement is true: *there is no nonracemic mixture of isomers for which the chirality function vanishes, being zero whatever the nature of the ligands may be.* We call a function of this type "qualitatively complete." In order to be able to construct such functions or to criticize a given theory on this line we have to get an idea of the set of all nonracemic mixtures. We need especially a concept referring to isomeric mixtures which can be called chiral or achiral, independently of the composition of the ligand assortment.

A linear combination of permutation operators of the symmetric group \mathcal{S}_n with positive real coefficients can be defined as an operator which, when applied to a mole of identical molecules, gives rise to a mixture of isomeric components, the coefficients being the mole ratios. Therefore we speak of an ensemble operator and call it chiral if the resulting mixture is nonracemic in case all ligands are different; otherwise it is called achiral. Because an ensemble operator is simultaneously an element of the group algebra it can be decomposed into components belonging to the irreducible representations $\Gamma_1 \Gamma_2 \dots$ of the group \mathcal{S}_n . Among these we can specify those which contain the chirality representation by means of character tables (characters for permutations representing rotations equal to +1 and for permutations representing reflections equal to -1), and simultaneously we can find a number z_r for each representation Γ_r which shows how often the chirality representation is contained. These properties tell us what we need for the components of an ensemble operator. All components belonging to representations with $z_r = 0$ are achiral, and all ensemble operators belonging to a representation with $z_r \neq 0$ can be decomposed into linear combinations of linearly independent components, where z_r^2 of them are chiral. A permutation operator can be defined as also acting on a chirality function, and all functions generated from one given chirality function by all permutation operators define a representation space of \mathcal{S}_n . Thus we find the mathematical criterion for qualitative completeness of a

chirality function. The result reads as follows: *a chirality function is qualitatively complete if, and only if, by applying the permutation operators of \mathcal{Q}_n we get a representation space for the representation*

$$\Gamma = \sum_r z_r \Gamma_r$$

where z_r are the numbers explained above.

Although the concept "qualitatively complete" is derived from properties related to isomeric mixtures it is relevant for the description of the chirality phenomenon on a homogeneous set of molecules also. That means that it is relevant for the phenomenon related to a molecule as well. The general properties of chirality functions due to their qualitative completeness, as well as mathematical proofs, are given elsewhere.⁴

We content ourselves with learning some of the consequences on the basis of special examples. For this purpose we use predominantly our first approximation method for qualitatively complete chirality functions. This method leads to chirality functions which are linear combinations of polynomials each of which belongs to one of the $\Sigma_r z_r$ representation spaces and is of lowest possible degree in ligand specific parameters. Thereby we need $\Sigma_r z_r$ different parameters for each ligand, and polynomials belonging to different representation spaces each containing exactly one of them. Sometimes we shall refer to the second method which results in a linear combination of chirality functions belonging to different representation spaces as well, but each of these is a linear combination of functions which depend on variables of as few as possible ligands.

Formulas according to both methods can be used as a semiempirical Ansatz which necessitate the determination of parameters or functions from experiments to predict the chirality observation on further molecules, or they can be used as a very useful frame theory for a quantum-mechanical treatment of a special kind of pseudoscalar measurements on the molecules of a given class.

If we forget both fields of application we are still left with a lot of information which is of use for discussing the chirality phenomenon for a class of molecules. This point of view is predominantly what we shall point out in the course of further discussions.

For the class of allene derivatives we find according to the first method the following qualitatively complete chirality function

$$\chi(l_1, l_2, l_3, l_4) = \epsilon(\lambda_1 - \lambda_2)(\lambda_3 - \lambda_4) + \epsilon'(\mu_1 - \mu_2)(\mu_2 - \mu_3)(\mu_3 - \mu_1) \times (\mu_4 - \mu_1)(\mu_4 - \mu_2)(\mu_4 - \mu_3)$$

where the arguments l_i represent now a set of real parameters (in our case λ_i and μ_i) specific for the ligands on correspondingly numbered skeleton sites, the coefficients ϵ and ϵ' equal +1 or -1. Contrary to our experience with the formerly used chirality function for this class we do not get a zero identity in case of the mixture shown in Figure 6 but

$$\frac{1}{3} \{ \chi(l_1, l_2, l_3, l_4) + \chi(l_1, l_3, l_4, l_2) + \chi(l_1, l_4, l_2, l_3) \} =$$

$$\epsilon'(\mu_1 - \mu_2)(\mu_2 - \mu_3)(\mu_3 - \mu_1)(\mu_4 - \mu_1)(\mu_4 - \mu_2)(\mu_4 - \mu_3)$$

which shows that the first polynomial vanishes and the second is reproduced, thereby exclusively describing the pseudoscalar observation on this isomeric mixture.

Taking the nonracemic mixture given in Figure 7 we get

$$\frac{1}{2} \{ \chi(l_1, l_2, l_3, l_4) + \chi(l_1, l_4, l_3, l_2) \} =$$

$$\frac{\epsilon}{2} \{ (\lambda_1 - \lambda_2)(\lambda_3 - \lambda_4) + (\lambda_1 - \lambda_4)(\lambda_3 - \lambda_2) \}$$

which shows the dependence of μ disappears. The same is true for a molecule if the ligands at position 2 and 4 of Figure 1 are equal. In this case we get

$$\chi(l_1, l_2, l_3, l_2) = \epsilon(\lambda_1 - \lambda_2)(\lambda_3 - \lambda_2)$$

From these statements we conclude that components belonging to different representation spaces represent physically independent phenomena. According to the first method they are expressed by different parameters which represent independent physical qualities, and can be investigated separately by measuring corresponding mixtures.

Now, what about the physical meaning of these different phenomena? To answer this question we relate again to our special example and compare it with a different class, the class of methane derivatives (Figure 8). The qualitatively complete chirality function according to the first method is

$$\chi(l_1, l_2, l_3, l_4) =$$

$$\epsilon'(\mu_1 - \mu_2)(\mu_2 - \mu_3)(\mu_3 - \mu_1)(\mu_4 - \mu_1)(\mu_4 - \mu_2)(\mu_4 - \mu_3)$$

From the comparison of both classes we learn that the second term in the formula for allene derivatives describes what is exclusively responsible for the class of methane derivatives. Therefore the relative weight of this component tells us to what extent we are justified in taking a given skeleton of D_{2d} symmetry as having the approximate symmetry of a regular tetrahedron. This certainly is a question which one may ask if confronted with a skeleton which has not precisely the higher symmetry, but nearly so. Our formula shows how to find the answer to this question, as it consists of related

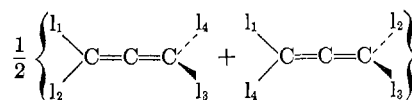


Figure 7.

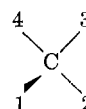


Figure 8.

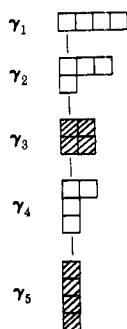


Figure 9.

components, which can be measured separately on corresponding isomeric mixtures.

Analogous situations are encountered whatever the chosen class may be. If our chirality function were not qualitatively complete, we would always miss certain contributions to any real pseudoscalar measurement. Related to a special kind of measurement, of course, it may happen that only some of the components give essential contributions. The question which component phenomena are of predominant significance cannot always be answered in advance by obvious criteria, and theories comprising only some of them may unfortunately be restricted just to components irrelevant for the measurements to be described. Quantum-mechanical treatments, for instance, which are characterized by a given order of perturbation theory may lead to a selection of components which turn out to be absurd from the point of the experimentalist's view.

We may visualize the essential background of our analysis and find its consequences in a pragmatic form by taking advantage of the fact that all irreducible representations of \mathcal{Q}_n are in a 1:1 correspondence with the so-called partition diagrams of the number n , diagrams which represent the various possibilities of decomposing the number n into a sum of numbers. Figure 9 shows the diagrams related to number 4. If we mark by shading all diagrams belonging to representations for which $z_r \neq 0$ we get, in the case of allene derivatives, the shadings of Figure 9. Let us be satisfied with the knowledge that it can be done generally according to well-defined rules. Because of the correspondence between irreducible representations and ensemble operators, the unshaded diagrams represent achiral ensemble operators, and for each shaded one we find at least one chiral ensemble operator.

Ensemble operators are just one concept related to the diagrams, but we find a completely different one which can also be assigned to the diagrams. Let the boxes be representations for ligands of the same kind as far as they belong to one horizontal row; then the diagrams represent compositions of ligand assortments as far as the numbers of equal ligands are concerned. We call such a composition a ligand partition. Furthermore, we shall call a ligand partition active if we can construct a chiral molecule by properly distributing the ligands on the skeleton sites; otherwise it will be called inactive. It can easily be proved that all ligand parti-

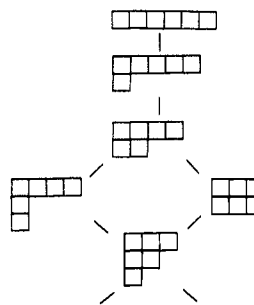


Figure 10.

tions represented by shaded diagrams are active, but we infer from our example that we may find further active partitions, because γ_3 , γ_5 , and γ_4 represent active partitions, and only γ_1 and γ_2 represent inactive ligand partitions.

One may get an idea how to find all active partitions by making use of a concept invented by Young⁵ (1901) which calls a diagram smaller than another one if, when descending from the top to the bottom, the length of its first different horizontal row of boxes is shorter. This definition allows an ordering of the diagrams along a chain as it was done for the number 4 in our figure. One finds the biggest at the top of the scheme and the smallest at the bottom. From the above example one feels inclined to suggest that a theorem can be found which shows that all diagrams not bigger than any of the shaded ones would represent active partitions, all others inactive partitions.

One can find examples which show that this suggestion is wrong. As we shall learn from the next section, having a rule like this would be of extreme advantage in winning further insight into the chirality phenomenon. Mathematicians have drawn many conclusions from Young's order concept, but these conclusions are not completely satisfying as far as the resulting statements are concerned. Therefore, even from the point of pure mathematics, a somewhat different structure concept would be desirable, a concept referring to a more fundamental structure of numbers which simultaneously allows one to state a theorem for ligand partitions different from but similar to the suggested one.

The Partition Lattice

We call a diagram, γ , smaller than another one, γ' , $\gamma \subset \gamma'$, if we can construct γ from γ' by pulling boxes from upper lines to lower ones. We supplement this definition by saying $\gamma \subset \gamma$ for each diagram γ . For all numbers $n \leq 5$ our definition turns out to be equivalent to that given by Young. But number 6 and higher numbers show even more fundamental differences (Figure 10). From Figure 10 we conclude: contrary to Young's definition which results in an "order," our definition results in a "half-order." The half-order can be proved to be a lattice; this means that for each pair of diagrams there exists a smallest bigger one and a biggest smaller one. The definition given

(5) A. Young, *Proc. London Math. Soc.* (1), **33**, 97 (1901).

above can be brought into another form by using partial sums o_i of the lengths ν_i of the i first horizontal rows $t = 1, 2, \dots, i$.

$$\begin{aligned} o_1 &= \nu_1 \\ o_2 &= \nu_1 + \nu_2 \\ o_3 &= \nu_1 + \nu_2 + \nu_3 \\ &\vdots \\ &\vdots \\ &\vdots \end{aligned}$$

Thereby we get $\gamma \subset \gamma'$ if, and only if, $o_i \leq o_i'$ for all i . Referring to this partition lattice we have proved the desired theorem which relates both to molecules and ensemble operators. *All partitions smaller than any shaded one, and only those, are active for molecules. All partitions smaller than a given shaded one, and only those, are active for corresponding chiral ensemble operators.* The latter statement says that nonracemic mixtures of isomers associated with a shaded diagram can be established by ligand partitions according to any of the smaller diagrams, but only those.

The consequences of these theorems are numerous. We shall give here just a selection of them. It can be concluded from the above theorems that related to a given diagram there exists no smaller diagram the first line of which is longer or the first column of which is shorter. Therefore we see that, within the set of all shaded diagrams, we can specify four numbers which should be characteristic for the chirality phenomenon of a given class, and we call them the chirality numbers of the class. These four numbers are given by the longest and the shortest first line and column of all shaded diagrams. Two of them are of outstanding interest, *i.e. the chirality order o , defined as the longest first line of all shaded diagrams; the chirality index u , defined as the shortest first column of all shaded diagrams.* It follows that *the chirality order defines the maximum number of equal ligands which occur in chiral molecules; the chirality index defines the minimum number of different kinds of ligands which are present in each chiral molecule.*

Let us go into further details with regard to the chirality order. It can be proved that $n - 3$ is a lower limit for the chirality order of all chiral classes. Furthermore, we can extrapolate our definition to get an interpretation for $o = n$ and $o = 0$. If o equals n , then molecules with ligands all equal to each other are chiral. Therefore $o = n$ defines classes with a chiral skeleton.

The case $o = 0$ can be interpreted as follows. If we have different ligands exclusively, which means each ligand is only equal to itself, then the molecule is already achiral. Therefore $o = 0$ defines achiral classes, *i.e., classes which do not contain chiral molecules (e.g., benzene derivatives).*

The chirality order concept shows its essence in connection with our second method mentioned above. The result of such an analysis can be formulated as a physical statement: If $o \neq 0$, then $n - o$ is the minimum

number of ligands which give additive contributions to a chirality observation by interaction. Therefore we distinguish five cases which show characteristic differences in theories related to them whatever kind they may be.

$o = n$. Molecules with ligands all equal to each other are chiral. Therefore the class has a chiral skeleton. A theory has to be concerned with describing the skeleton chirality.

$o = n - 1$. Molecules with $n - 1$ identical ligands, but not more, are chiral. Therefore pseudoscalar observations may be described as additional contributions from single ligands (octant, quadrant, sector rules).

$o = n - 2$. We may describe a chirality observation as a superposition of contributions from pairs of ligands (principle of pairwise interactions: Kauzmann, Clough, and Tobias^{6a}).

$o = n - 3$. We need at least three ligand interactions (e.g., methane derivatives).

$o = 0$. The class is achiral by definition.

It should be emphasized that the minimum number of interacting ligands mentioned in the above statements ensures that we get a component of the qualitatively complete chirality function different from zero. Corresponding theorems ensuring the qualitative completeness can also be derived from the partition lattice.⁴ Most of the known theories are found to be restricted to the component characterized by the chirality order.

We wish to exemplify the use of our structural insight for a criticism of a quantum-mechanical theory concerning a special class. To describe the optical rotatory power for the class of methane derivatives we take the central carbon atom with four directed bonds according to T_d symmetry as skeleton and allow ligands which have, at least with regard to their averaged conformations, the C_{3v} symmetry of the radial bonds. One sees from the maximum number of equal ligands present in chiral molecules that $o = 1$, and therefore $n - o = 3$, which means that we need three-ligand interactions. A quantum-mechanical treatment on the basis of multipole interactions between ligands as worked out by Kirkwood^{6b} for this class is characterized by the order of perturbation theory. The first-order perturbation gives only interactions between two ligands; the second order is very complicated so that one is inclined to apply only first-order perturbation treatment. This, however, cannot describe, as we have shown, a chirality observation for the above-defined class of molecules. Therefore, if we want to keep the first-order perturbation, as Kirkwood does, we are forced to weaken the above-mentioned conditions. That means, for instance, that we have to treat deviations from tetrahedron-like bond angles or a ligand conformation or a set of ligand conformations which has not the averaged symmetry C_{3v} . This change of condition leads to a situation which can be characterized by a higher chirality order, and therefore two ligand interactions are enough

(6) (a) W. Kauzmann, F. B. Clough, and I. Tobias, *Tetrahedron*, **13**, 57 (1961); (b) J. G. Kirkwood, *J. Chem. Phys.*, **5**, 479 (1937).

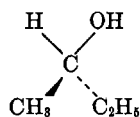


Figure 11.

to derive a chirality function. This chirality function, however, is not qualitatively complete: one neglects thereby a component which is due to an ensemble of conformers which has the above-mentioned symmetry properties. The neglected component would describe an additional part for the chirality observation, even if the real average of conformations has not the C_{3v} symmetry. Kirkwood's treatment gives a contribution which refers to the described deviations only, and his formula becomes zero if the deviations vanish. This may be good for the special molecule he treats (Figure 11) as far as the mentioned deviations are big enough to give the major part of the observed phenomena, but this should be tested. Generally this procedure is questionable, because we have a first-order effect with regard to the deviations which is described by first-order perturbation, but a zero-order effect in the deviations also which can be described only by the second-order perturbation.

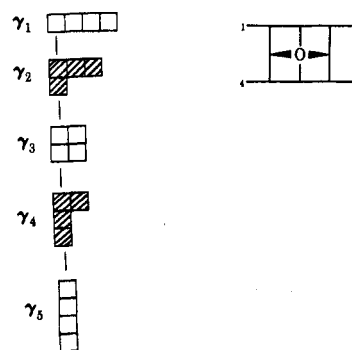
A generalized quantum-mechanical theory including second-order perturbation⁷ confirms what we have pointed out here. It leads to the physical interpretation of all quantities which occur in both our methods and are left undetermined there. It also allows an interesting comparison with Kirkwood's theory and other theoretical treatments of the optical rotatory power for the class of methane derivatives.

In conclusion: the algebraic structure we have explained here provides many systematic aspects related to the chirality phenomenon itself and to theories concerned with special chirality observations. More remarks which refer exclusively to the use of the given theorems and formulas for experimental purposes without the need of further reference to quantum mechanics are found in the next section.

Experimental Remarks

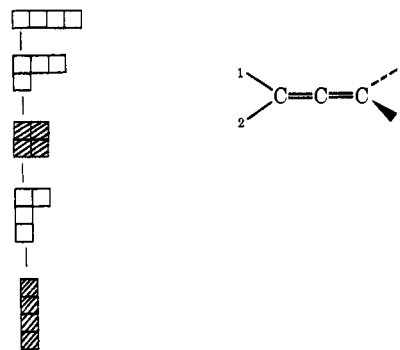
We may use the formulas according to one of our methods, let us say the first method, as a semiempirical procedure for discussing the optical rotatory power as an example. This is possible because the number of different chiral molecules grows more rapidly than the number of experiments one needs to determine the parameters of involved ligands.

In the case of classes where qualitatively complete formulas become very complicated, we may restrict ourselves to subclasses for which these formulas become simpler. These subclasses will be found to be just those which are interesting from the point of chemistry. The situation becomes clear if we remember that restrictions with regard to ligand partitions lead generally to the fact that certain components of qualitatively complete chirality functions vanish. Therefore condi-



$$\chi(l_1, l_2, l_3, l_4) = \lambda_1 - \lambda_2 + \lambda_3 - \lambda_4 + (\mu_1 - \mu_3)(\mu_2 - \mu_4)(\mu_1 - \mu_2 + \mu_3 - \mu_4)$$

Figure 12.



$$\chi(l_1, l_2, l_3, l_4) = \epsilon(\lambda_1 - \lambda_2)(\lambda_3 - \lambda_4) + \epsilon'(\mu_1 - \mu_2)(\mu_2 - \mu_3)(\mu_3 - \mu_1)(\mu_4 - \mu_1)(\mu_4 - \mu_2)(\mu_4 - \mu_3)$$

Figure 13.

tions like "not more than a certain number of different kinds of ligands" or "not less than a certain number of equal ligands, say hydrogen atoms" lead to simplified formulas which can be read from the partition lattice. With reference to subclasses defined in this way these simplified formulas are then qualitatively complete.

As a first example we give the qualitatively complete chirality function according to the first method for adamantanone derivatives according to Figure 12. If we restrict ourselves to molecules with ligand partitions not smaller than γ_4 , which means to molecules having not more than two different kinds of ligands, then the polynomial in μ vanishes and the λ polynomial, therefore, is qualitatively complete. In other words, for this subclass the quadrant rule is correct.

As a second example we take again the class of allene derivatives (see Figure 13). If we restrict ourselves to molecules with less than four different kinds of ligands then we are left with the λ polynomial.

Let us try a discussion *in extenso* for measurements of the rotation angle at the sodium D line for allene derivatives. We may hope that even for derivatives having four different ligands a description by the λ component is satisfying. This can be expected because we may consider the accumulated double bonds essentially as being the skeleton influenced by ligands in their respective positions, and this situation is very far from being a regular tetrahedron. To be sure, though, that experimentally determined λ values are not falsified by the

neglect of the μ component, we use for its determination exclusively compounds with two equal ligands.

To test our procedure we need more different compounds than the number of kinds of ligands involved. The number $2p$ of different chiral molecules which can be built up from a given number N of different kinds of ligands grows according to the formula

$$p = \frac{1}{8}N(N-1)\{(N-2)(N+1)+4\}$$

the above relation can be derived according to methods originated by Pólya⁸ and generalized by us.⁹ Therefore r gives the number of different pairs of enantiomeric molecules. Table I gives a quantitative idea of this relation.

Because we could not find a known variety of compounds which is bigger than the number of constituting ligands, we will refer to some new compounds and corresponding measurements which will be published in detail shortly.¹⁰ Table II gives the measured rotation angles for compounds which are used for the determination of ϵ and the λ values in Table III. The calculated rotation angles in Table II differ from the experimental ones just by round-off errors of the λ values. Because

Table I

N	1	2	3	4	5	10
p	0	1	6	21	55	1035

Table II

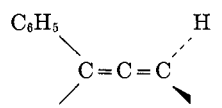
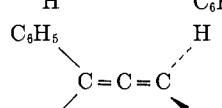
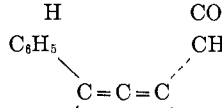
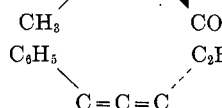
Compound	Ref	Exptl	Calcd
	11	+1020	+1018
	12	+420	+421
	10	+149	+148
	10	+41	+41

Table III

ϵ	= 1
$\lambda(\text{H})$	= 0
$\lambda(\text{CH}_3)$	= 7.2
$\lambda(\text{C}_2\text{H}_5)$	= 11.2
$\lambda(\text{C}_6\text{H}_5)$	= 31.9
$\lambda(\text{COOH})$	= 13.2

(8) G. Pólya, *Acta Math.*, **68**, 145 (1937).

(9) E. Ruch, W. Hässelbarth, and B. Richter, *Theor. Chim. Acta*, **19**, 288 (1970).

(10) G. Kresze, W. Runge, and E. Ruch, *Chem. Ber.*, in preparation.

(11) J. M. Walbrick, J. M. Wilson, and W. M. Jones, *J. Amer. Chem. Soc.*, **90**, 2895 (1968).

(12) K. Shingu, S. Hagishita, and M. Nakagawa, *Tetrahedron Lett.*, 4371 (1967).

Table IV

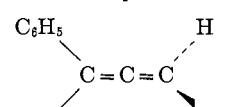
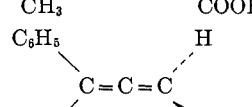
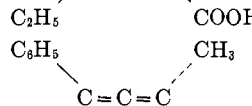

Compound	Ref	Exptl	Calcd
	12	+318	+326
	12	+280	+273
	10	+125	+124
	10	+51	+49

Table V

$\lambda(i\text{-Pr})$	≈ 16.1	$\lambda(\alpha\text{-Np})$	≈ 50.4
$\lambda(i\text{-Bu})$	≈ 18.2	$\lambda(n\text{-Pr})$	≈ 14.9
$\lambda(\text{COOEt})$	≈ 13.6	$\lambda(\text{COOMe})$	≈ 13.4
$\lambda(n\text{-Bu})$	≈ 18.2		

λ values can be determined by experiments apart from a common additive constant, we have chosen $\lambda(\text{H}) = 0$. Table IV contains experimental and predicted angles which are determined using the parameters of Table III, thus furnishing a test for the applied formula.

The correspondence seems convincing: it shows that the "shortened Ansatz" consisting of the λ polynomial is completely satisfactory, and this gives impetus to the determination of λ parameters for further ligands. Some, derived from measurements not shown here, are given in Table V. Brewster¹³ has also discussed the optical rotatory power for allene derivatives on the basis of ligand-specific parameters. Experiments for other classes would be of interest, both from the experimentalist's point of view and for testing corresponding formulas. Of course, after having made sure that formulas are quantitatively satisfactory, the interest becomes directed to compounds which are in disagreement because just deviations from rules necessitate explanations which give additional information concerning, *e.g.*, the influences of the solvent, hydrogen bonds, or skeleton distortions, etc.

Finally we should like to emphasize that the significance of the algebraic theory, sketched in this Account, is not primarily based on the quantitative aspects of formulas according to one or both of our methods. Rather it should be judged from the point of insight into the chirality phenomenon itself, its decomposition into components, and the consequence connected therewith. Moreover, questions which have arisen in this context refer to pure mathematics and seem to lead to interesting aspects in this field.

I express my heartiest thanks to Professors I. Dunnitz and H. Simmons for many helpful suggestions on style and grammar.

(13) J. H. Brewster, *Top. Stereochem.*, **2**, 1 (1967).